A public framework for calculating the forces between magnets and multipole arrays of magnets

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***Abstract*—In this paper we present a public framework in Matlab for calculating forces between magnets and between multipole arrays of magnets. The multipole arrays are situated as pairs in opposition and the force vs. displacement characteristics calculated based on a forces superposition of the magnets that compose the arrays. The software framework provides a convenient method of calculating magnetic forces and allows useful comparisons between a variety of multipole configurations. Design optimisation may also be performed as the abstractions of the software permit easy iteration of high-level variables of systems of interest. The code is public and may be re-used and modified by anyone, hence providing a useful service to researchers in the field who wish to perform such calculations as described above. Collaboration is encouraged to improve and extend the software, yielding benefits to the research community as a whole.**

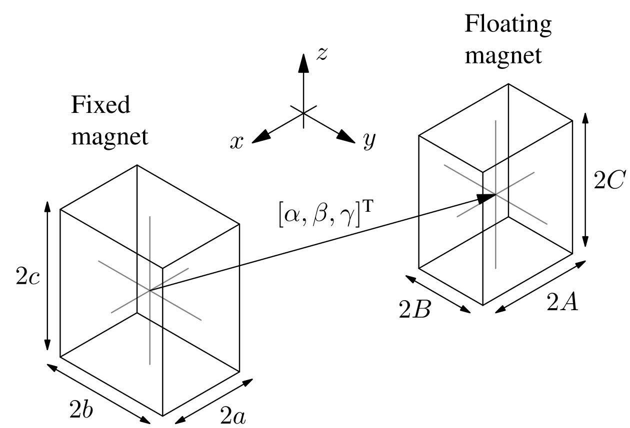


Fig. 1: Depiction of the geometry for the two-magnet system.

I. Introduction

In 1984, Akoun and Yonnet published their expression for calculating the forces between two parallel cuboid magnets with vertical magnetisation. Much more recently, Yonnet and Allag (2009) published the analogous formulation for magnets with orthogonal magnetisation. Equivalent expressions have also been published by Janssen et al. (2009).

With these works, it is possible to calculate the force between any two cuboid magnets with arbitrary magnetisation directions (although the magnets themselves must have parallel edges; no magnet rotation is permitted). However, for a new researcher attempting to use this theory, some non-insignificant time investment must be spent (a) implementing the theory in the programming language of their choice, (b) providing a convenient interface such that all the necessary coordinate system transformations for arbitrary magnetisations are handled automatically, and finally (c) testing that it is correct.

In this paper we present a public framework of verified and tested Matlab code to calculate the forces between arbitrarily magnetised magnets. In addition, convenient functions are provided to easily calculate the forces between arbitrary multipole arrays of magnets. Anyone may download this code to use it, modify it and re-distribute it freely. With further work and hoped collaboration, we plan to add support for more programming languages in the future. We believe that it is important to share the tools of one's research to help accelerate the progress of new researchers in the field and to help the reproducibility and verification of published results.

II. Details of the code

The implementation of the theory discussed in this paper is located at http://github.com/wspr/magcode. The source of the Matlab code, which is fully documented in a literate programming style (Knuth 1984), and the corresponding Matlab functions for calculating the forces between magnets and arrays of magnets, are contained within the 'matlab/' subdirectory; examples of using the code, including for the plots shown in this paper, are contained within 'examples/'.

III. Forces between magnets

The expressions for calculating the forces themselves will not be shown again here as they have recently been reprinted (Allag, Yonnet, and Latreche 2009). The Allag et al. expressions are used instead of the Janssen et al. expressions as the former are slightly simpler mathematically than the latter. The geometry of the two-magnet system is shown in Figure 1, in which the magnets have side lengths $\boldsymbol{s} = [2a, 2b, 2c]^{\mathrm{T}}$ and $\boldsymbol{S} = [2A, 2B, 2C]^{\mathrm{T}}$ respectively and the distance between their centres is given by $\boldsymbol{d} = [\alpha, \beta, \gamma]^{\mathrm{T}}$. The calculations always assume that the first magnet is fixed and force is acting on the second magnet. The signs must be reversed to obtain the forces acting on the first magnet.

Akoun and Yonnet (1984) provide the force expressions for magnets with vertical magnetisations. This force is denoted herein as $\boldsymbol{F}_{z,z}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_1, J_2)$ as a function of the magnet sizes, the distance between them, and their magnetisation magnitudes J_1 and J_2. Allag, Yonnet, and Latreche (2009) provide the force expressions for the first magnet with vertical magnetisation and the second magnet with magnetisation in the horizontal y-direction. This force is denoted herein as $\boldsymbol{F}_{z,y}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_1, J_2)$.

The force between a vertically-magnetised magnet and one with magnetisation in the horizontal x-direction can be calculated by applying a rotational transformation to $\boldsymbol{F}_{z,y}$ around the z-axis. That is,

$$\boldsymbol{F}_{z,x}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_1, J_2) = \mathbf{R}_z\left(-\tfrac{\pi}{2}\right) \boldsymbol{F}_{z,y}(\boldsymbol{s}_{z,x}, \boldsymbol{S}_{z,x}, \boldsymbol{d}_{z,x}, J_1, J_2), \quad (1)$$

where $\boldsymbol{s}_{z,x} = \operatorname{abs}\big(\mathbf{R}_z\left(\tfrac{\pi}{2}\right) \boldsymbol{s}\big)$, $\boldsymbol{S}_{z,x} = \operatorname{abs}\big(\mathbf{R}_z\left(\tfrac{\pi}{2}\right) \boldsymbol{S}\big)$, and $\boldsymbol{d}_{z,x} = \mathbf{R}_z\left(\tfrac{\pi}{2}\right) \boldsymbol{d}$, for which $\operatorname{abs}(\cdot)$ is the *element-wise* absolute value function and $\mathbf{R}_z(\theta)$ is the rotation matrix around the z-axis.

Using the force expressions $\boldsymbol{F}_{z,x}$, $\boldsymbol{F}_{z,y}$, and $\boldsymbol{F}_{z,z}$ in superposition allows the force to be calculated between a vertically magnetised magnet and another magnet with arbitrary magnetisation direction. By applying coordinate system transformations to these expressions, arbitrary magnetisation directions can be achieved for the first magnet as well.

For horizontal x-direction magnetisation of the first magnet

$$\boldsymbol{F}_{x,\{x,y,z\}}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_1, J_2) = \mathbf{R}_y\left(\tfrac{\pi}{2}\right) \boldsymbol{F}_{z,\{z,y,x\}}(\boldsymbol{s}_x, \boldsymbol{S}_x, \boldsymbol{d}_x, J_1, J_2), \quad (2)$$

where $\boldsymbol{s}_x = \operatorname{abs}\big(\mathbf{R}_y\left(-\tfrac{\pi}{2}\right) \boldsymbol{s}\big)$, $\boldsymbol{S}_x = \operatorname{abs}\big(\mathbf{R}_y\left(-\tfrac{\pi}{2}\right) \boldsymbol{S}\big)$, $\boldsymbol{d}_x = \mathbf{R}_y\left(-\tfrac{\pi}{2}\right) \boldsymbol{d}$, and $\mathbf{R}_y(\theta)$ is the rotation matrix around the y-axis.

Similarly, for horizontal y-direction magnetisation of the first magnet

$$\boldsymbol{F}_{y,\{x,y,z\}}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_1, J_2) = \mathbf{R}_x\left(-\tfrac{\pi}{2}\right) \boldsymbol{F}_{z,\{x,z,y\}}(\boldsymbol{s}_y, \boldsymbol{S}_y, \boldsymbol{d}_y, J_1, J_2), \quad (3)$$

where $\boldsymbol{s}_y = \operatorname{abs}\big(\mathbf{R}_x\left(\tfrac{\pi}{2}\right) \boldsymbol{s}\big)$, $\boldsymbol{S}_y = \operatorname{abs}\big(\mathbf{R}_x\left(\tfrac{\pi}{2}\right) \boldsymbol{S}\big)$, $\boldsymbol{d}_y = \mathbf{R}_x\left(\tfrac{\pi}{2}\right) \boldsymbol{d}$, and $\mathbf{R}_x(\theta)$ is the rotation matrix around the x-axis.

Given the results of the afore-referenced papers by Yonnet et al. and Equations (1), (2), and (3), the force between two magnets of arbitrary magnetisation can be written as

$$\boldsymbol{F}(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, \boldsymbol{J}_1, \boldsymbol{J}_2) = \sum_{i,j\in\{x,y,z\}^2} \boldsymbol{F}_{i,j}\big(\boldsymbol{s}, \boldsymbol{S}, \boldsymbol{d}, J_{1_i}, J_{2_j}\big) \quad (4)$$

where

$$\boldsymbol{J}_1 = [J_{1_x}, J_{1_y}, J_{1_z}]^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{J}_2 = [J_{2_x}, J_{2_y}, J_{2_z}]^{\mathrm{T}}. \quad (5)$$

The theory to evaluate Equation (4) is implemented in the Matlab function ‘magnetforces’.

A. Examples

The verification curve of the results of Akoun and Yonnet (1984), calculated using (4) with the ‘magnetforces’ function, is shown in Figure 2 for the following system:

$$\begin{aligned} \boldsymbol{s} &= [20\,\mathrm{mm}, 12\,\mathrm{mm}, 6\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{S} &= [12\,\mathrm{mm}, 20\,\mathrm{mm}, 6\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{d} &= [-4\,\mathrm{mm} + \delta, -4\,\mathrm{mm}, 8\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{J}_1, \boldsymbol{J}_2 &= [0, 0, 0.38\,\mathrm{T}]^{\mathrm{T}} \end{aligned} \quad (6)$$

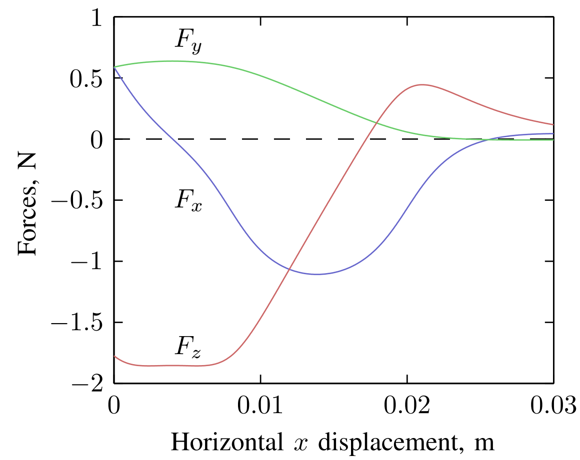


Fig. 2: Reproduction of the results shown by Akoun and Yonnet (1984).

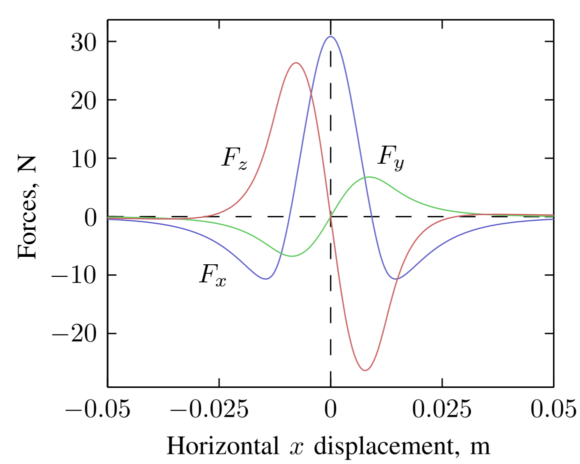


Fig. 3: Reproduction of the results of Janssen et al. (2009).

The verification curve of the results shown by Janssen et al. (2009) is shown in Figure 3 for the following system:

$$\begin{aligned} \boldsymbol{s} &= [10\,\mathrm{mm}, 26\,\mathrm{mm}, 14\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{S} &= [14\,\mathrm{mm}, 26\,\mathrm{mm}, 10\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{d} &= [\delta, -8\,\mathrm{mm}, 15\,\mathrm{mm}]^{\mathrm{T}} \\ \boldsymbol{J}_1 &= [0, 0, 1\,\mathrm{T}]^{\mathrm{T}}, \qquad \boldsymbol{J}_2 = [1\,\mathrm{T}, 0, 0]^{\mathrm{T}} \end{aligned} \quad (7)$$

The Matlab code to generate Figures 2 and 3 is located in the file ‘examples/magnetforces_example.m’. The original verification curves cited above were compared, and found to agree, with experimentally measured values.

B. Stiffness between magnets

The stiffnesses in each direction can also be calculated through an analytical differentiation of the force expressions given in this section. See the code documentation for further details.

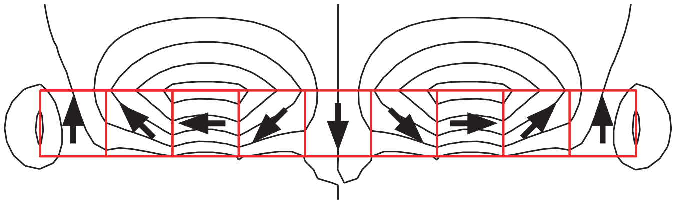


Fig. 4: Magnetic field lines for a linear multipole array with 45° magnetisation rotations. The single-sided nature of the magnet field is evident.

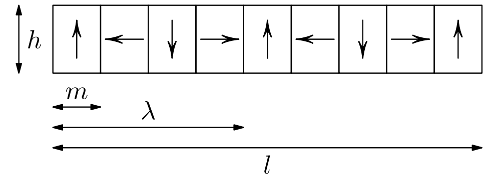


Fig. 5: Geometry of a linear multipole array with 90° magnetisation rotations. This array contains two wavelengths of magnetisation with an end magnet for symmetry.

IV. Forces between multipole magnet arrays

A classical multipole or Halbach array is a linear array of magnets stacked to approximate a single magnet with sinusoidal magnetisation, first analysed in the '70s (Halbach 1981; Shute et al. 2000). Multipole arrays have been analysed for a variety of force-producing applications; only a small selection are included in the bibliography here (Lee, Lee, and Gweon 2004; Robertson, Cazzolato, and Zander 2005; Rovers et al. 2009). One advantage of using multipole arrays is to focus the magnetic field on one side of the array, such to increase the forces exerted by which magnetic field on one side of the array and to reduce or eliminate any need for magnetic shielding on the reverse side. The magnetic field produced by one such multipole array is shown in Figure 4, which was generated with finite element analysis, in which the single-sided nature of the magnetic field can clearly be seen.

Having expressed in Section III the forces between two magnets with arbitrary magnetisation, it becomes straightforward in theory to use this expression iteratively over an array of magnets with varying magnetisation strengths and/or directions. The force between two arrays is simply the superposition of every combination of forces between the individual magnets in each array. For example, for two five-by-five magnet arrays (such as the planar Halbach array described later and shown in Figure 8), a total of $(5 \times 5)^2 = 625$ calculations must be made, each of which containing up to 3×3 individual force calculations for magnetisations in each direction.

The iteration over each combination of magnets is abstracted to allow us to simplify the code necessary to express a variety of multiple array configurations. This facilitates easy comparisons between different designs, including multipole arrays that follow different magnetisation patterns than a Halbach array as well as planar arrays with magnetisations that vary in all three cartesian directions. Examples of such arrays will be shown in Section V. The code implementation to calculate the forces between generic multipole arrays is contained within the Matlab function 'multipoleforces'.

A. *Halbach-like multipole arrays*

A multipole array can be uniquely defined in terms of several sets of variables. The simplest such description is:

- Size of each magnet $[m, d, h]^{\mathrm{T}}$,
- Number of magnets N, and
- Magnetisation direction of the first magnet ϑ_0 and rotation between successive magnets ϑ_i.

Other variables that can also be used to describe the array are: (assuming an array aligned with the x-axis)

- Length of the array $l = mN$,
- Number of magnets per wavelength $M = 2\pi/\vartheta_i$,
- Wavelength of magnetisation $\lambda = mM$, and
- Number of wavelengths $W = [N-1]/M$.

Figure 5 illustrates the relationship between magnet length, array length, and wavelength for an example linear array. The wavelength of magnetisation is the length required to achieve, with successive magnets, a full rotation of magnetisation direction.

Note the presence in Figures 4 and 5 and in general of an 'end magnet' that adds symmetry to the discretisation of the magnetisation. This extra magnet is necessary to balance the forces in the horizontal direction.

Provided that enough information is specified and it is internally consistent, the Matlab implementation for calculating array forces can accept any combination of the variables listed above when defining the geometry of each array. Rather than explicitly enumerating the location of each magnet and the direction of its magnetisation, the implementation requires just an axis with which to align the array and the facing direction of its 'strong' side.

B. *Example of linear multiple arrays*

Allag et al. Allag, Yonnet, and Latreche (2009) calculated the forces between two five-magnet multipole arrays with 90° rotation between successive magnets. This system is shown in Figure 6, and the parameters used for the simulation were:

- Arrays aligned along y and facing vertically $\pm z$.
- $[m, d, h]^{\mathrm{T}} = [0.01\,\mathrm{m}, 0.01\,\mathrm{m}, 0.01\,\mathrm{m}]^{\mathrm{T}}$.
- Total number of magnets $N = 5$ and number of magnets per wavelength of magnetisation $W = 4$.
- Displacement between arrays $\boldsymbol{d} = [0, \delta, 0.015\,\mathrm{m}]^{\mathrm{T}}$.

Their results are reproduced in Figure 7. This reproduction may be found in the file 'examples/multipole_example.m'. This is a good example of the convenience of using this framework for calculating results of this kind. Of the fifty lines of code in 'multipole_example.m', only twenty lines of code are necessary to set up the system parameters and calculate the forces; the rest of the file consists of comments, whitespace, and producing the actual figure itself.

V. Planar multipole arrays

Planar multipole arrays consist of a two dimensional stack of magnets that vary in magnetisation as a function of their

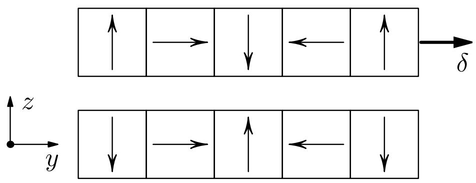


Fig. 6: Multipole system composed of $10\,\mathrm{mm}$ cube magnets for $\delta = 0$. Forces in the y- and z-directions as δ varies are shown in Figure 7.

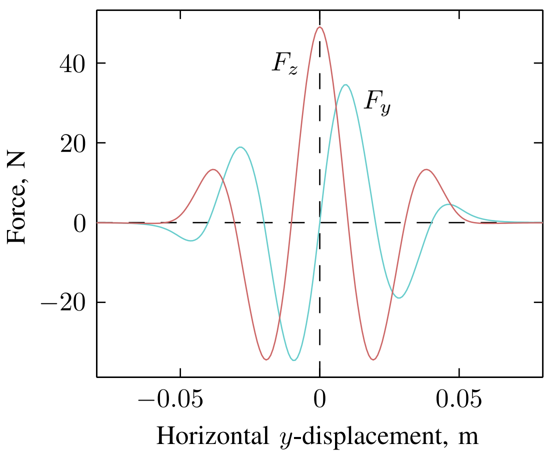


Fig. 7: Reproduction of Allag et al.'s results Allag, Yonnet, and Latreche (2009). The forces in the x-direction are zero.

position. Kim (1997, Appendix A) has proposed a planar multipole array based on the magnetisation of two superimposed linear multipole arrays. Such a system, called here a 'planar Halbach' array, is shown in Figure 8, with five magnets per side and 90° magnetisation rotation in both the x–z and y–z planes between successive magnets.

Moser et al. (2002), Rovers et al. (2009), and Janssen, Paulides, and Lomonova (2009*a*) (the latter two with associated publications) have examined the idea of a 'quasi-Halbach' planar multipole array in which all magnetisation directions are restricted to one of the orthogonal directions of the axes

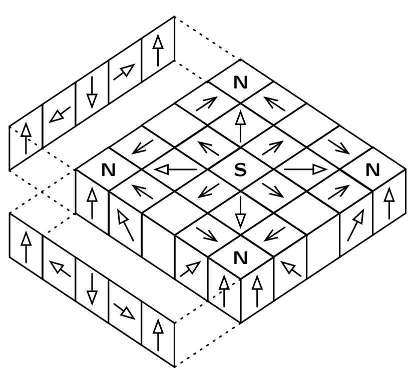
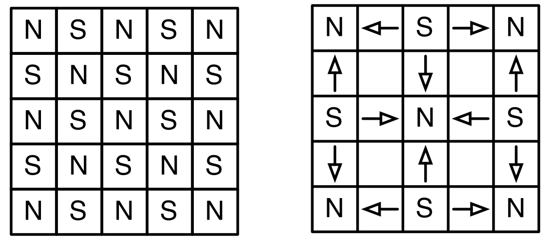


Fig. 8: A planar Halbach array, facing up, with magnetisation directions as the superposition of two orthogonal linear Halbach arrays. Non-filled arrowheads denote diagonal magnetisation and empty magnets have zero magnetisation.



(a) Patchwork array. (b) Quasi-Halbach array.

Fig. 9: Two planar multipole arrays, facing 'towards the reader'. Note the areas of zero magnetisation in the quasi-Halbach array.

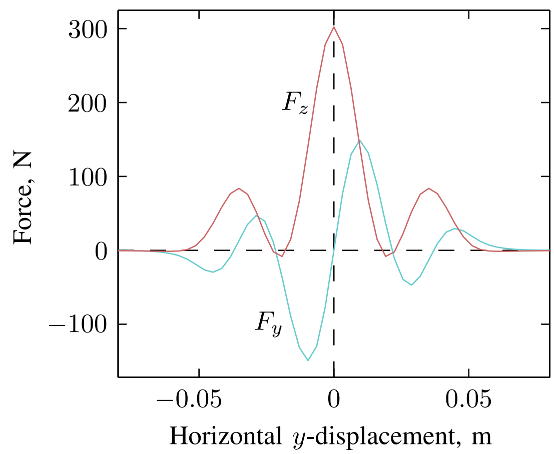


Fig. 10: Force vs. horizontal displacement results for two planar Halbach arrays such as shown in Figure 8. The x-forces are all zero.

(i.e., no diagonal magnetisations). Another planar multipole array, simpler again, is the 'patchwork' array, which alternates between positive and negative vertical magnetisation between successive magnets in both directions. The patchwork and quasi-Halbach arrays are shown respectively in Figure 9.

A. *Example of planar Halbach array forces*

Force characteristics for the quasi-Halbach array have been shown recently Janssen, Paulides, and Lomonova 2009*a* but the planar Halbach array has not yet been analysed. Since the planar Halbach array uses magnetisations in directions non-orthogonal to the axes, the necessary calculations are more complex than for the quasi-Halbach array. For two facing planar Halbach arrays, Figure 10 shows results of y- and z-force vs. horizontal y-displacement with arrays composed of $10\,\mathrm{mm}$ cube magnets and $15\,\mathrm{mm}$ vertical displacement between the array centres. Each magnet of each array (except those with zero magnetisation) has a magnetisation of $1\,\mathrm{T}$. The Matlab code to generate Figure 10 is located in the file 'examples/planar_multipole_example.m'.

Note the differences in the shape of the force curves between Figures 7 and 10 for linear and planar arrays respectively; in the planar Halbach case, the vertical force is largely positive even as the arrays shift into their attractive zone, whereas the linear multipole forces are more symmetrical and become significantly negative under a similar displacement. A more detailed comparison between various configurations of Halbach arrays is currently under investigation and will be reported at a later date.

B. *Example comparion between planar arrays*

When pairs of arrays are faced in opposition, a vertical force is produced between them that can be larger than the corresponding force between two equivalently-sized magnets with homogeneous magnetisation. Each array configuration has a different magnetic field pattern and has a different force/displacement profile. It is now possible to evaluate between and optimise the array configurations based on the design requirements for which they are required.

The vertical force vs. vertical displacement characteristics of each array discussed (linear Halbach, planar Halbach, patchwork, and quasi-Halbach) are compared in Figure 11, with the forces between two single magnets (of the same size as the arrays) included for comparison. The linear Halbach, planar Halbach, and quasi-Halbach have all been chosen to have a single wavelength of magnetisation with an end magnet for symmetry. Each array has the same outer dimensions of $50\,\mathrm{mm} \times 50\,\mathrm{mm}$, thickness $10\,\mathrm{mm}$, and composed of magnets with magnetisation $1\,\mathrm{T}$, and each magnet is either cube-shaped or, for the linear array, has a square cross-section. The Matlab code to generate Figure 11 is located in the file 'examples/planar_compare.m'.

It is interesting that the linear array exhibits the greatest force for a given displacement; this can be explained by the fact that the planar array and the quasi-Halbach array both have regions of zero magnetisation; this degrades their load-bearing ability since there is less inherent magnetic energy in each. These results indicate that the linear array is the most suitable choice for bearing vertical loads; not only are the forces stronger, but the magnetisation arrangement (and hence construction of the array) is simpler as well. It is also notable that despite the much simpler magnetisation pattern of the quasi-Halbach array compared to the planar Halbach array, the results for these two are quite similar, especially at small displacements.

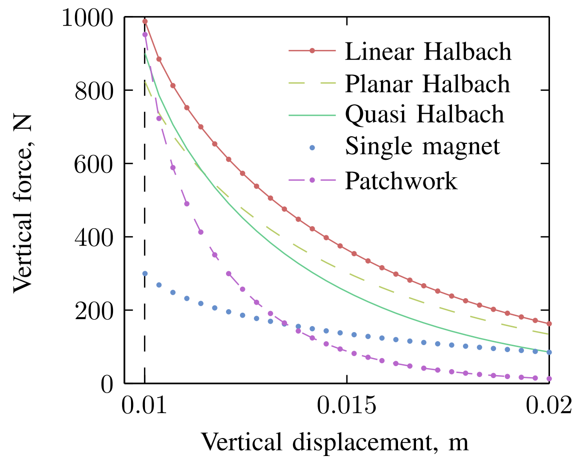


Fig. 11: Vertical forces vs. vertical displacement between the array centres to compare the load-bearing ability of a range of magnet arrays. The dashed vertical line indicates the displacement at which the array faces are in contact.

VI. Future work

The code presented here currently only contains algorithms for calculating forces and stiffnesses between cuboid magnets without rotation. Charpentier and Lemarquand (1999) present expressions for two components of the force between cuboid magnets with rotation around one axis (also see their related publications around the same time), but there is no general solution known without using semi-numerical methods (Charpentier and Lemarquand 2001).

Agashe and Arnold (2008) and Nagaraj (1988) have also calculated forces between cuboid magnets as well as between cylindrical magnets, which latter have more complex solutions due to the presence of elliptic integrals. Janssen, Paulides, and Lomonova (2009*b*) have derived expressions to calculate the analytical forces between pyramidal-frustrum shaped magnets, which expressions have not yet been published at time of writing. Lee et al. (2006) and Cho, Im, and Jung (2001) present different planar arrays using block magnets with trapezoidal and triangular cross-sections, Marble (2008) propose a nonlinear magnetisation function to maximise the field strength for a linear array, and Yan et al. (2006) calculate torque for spherical multipole arrays with 'dihedral cone'-shaped magnets. And there are a wide variety of cylindrical multipole arrays in use for bearing and brushless motor applications (Zhu and Howe 2001).

As such, there is great scope for generalising the approach used in the presented software framework. The 'magnetforces' function can be extended to calculate torques as well and incorporate different magnet geometries and rotations, and the 'multipoleforces' function can be extended to abstract more configurations of linear and planar multipole arrays (including three dimensional stacks of magnets). The work introduced here is a very humble beginning to incorporating as much of the cited work above (and this is certainly an incomplete list) into a unified framework for conveniently and confidently calculating the dynamics of permanent magnet–based systems.

VII. Conclusion

We have introduced a tested Matlab framework, suitable for public use, for calculating the forces between magnets and between multipole arrays of magnets. This framework adds an abstraction to the process of performing the calculations, permitting fast and easy design iterations based on its results. Early results have allowed us to compare the force/displacement curves for a variety of planar arrays. We hope that others in the community will find this code useful.

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